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Connected operators for the totally asymmetric exclusion process

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Abstract
We elucidate the structure of the hierarchy of the connected operators that commute with the Markov matrix of the totally asymmetric exclusion process. We derive the combinatorial formula for the connected operators that was conjectured in our previous work.

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1. Introduction

The asymmetric simple exclusion process (ASEP) is a lattice model of particles with hard core interactions that has become a paradigm in the field of non-equilibrium statistical mechanics (for reviews, see e.g., Derrida (1998), Golinelli and Mallick (2006)). In a recent work (Golinelli and Mallick 2007), we used the algebraic Bethe ansatz to construct the transfer matrix of the totally asymmetric exclusion process (TASEP), and obtained a hierarchy of local connected operators that commute with the TASEP Markov matrix. We conjectured an explicit combinatorial formula for these operators. In the present work, we prove analytically this formula. In section 2, useful algebraic results are briefly reviewed so that this work can be read in a fairly self-contained manner.

2. The TASEP algebra

The TASEP on a periodic 1-d ring with L sites evolves according to the following dynamics: during the time interval $[t, t + \mathrm{d}t]$, a particle on a site i jumps with probability $\mathrm{d}t$ to the neighbouring site $i+1$, if this site is empty (*exclusion rule*). This dynamics is entirely encoded in a $2^L \times 2^L$ Markov matrix $M = \sum_{i=1}^{L} M_i$, which can be written as a sum of the local jump operators M_i satisfying a set of algebraic relations

$$M_i^2 = -M_i, \tag{1}$$

$$M_i M_{i+1} M_i = M_{i+1} M_i M_{i+1} = 0, \tag{2}$$

$$[M_i, M_j] = 0 \qquad \text{if} \quad |i-j| > 1. \tag{3}$$

Besides, we have periodic boundary conditions on the ring: $M_{i+L} = M_i$. Any product of the M_i's will be called a *word*. The *length* of a given word is the minimal number of operators M_i required to write it. A *reduced* word cannot be simplified further.

Consider any word W and call $\mathcal{I}(W)$ the set of indices i of the operators M_i that compose it (indices are enumerated without repetitions). We remark that, if W is not annihilated by application of rule (2), the simplification rules (1), (3) do not alter the set $\mathcal{I}(W)$, i.e., these rules do not introduce any new index or suppress any existing index in $\mathcal{I}(W)$.

A *simple word* of length k is defined as a word $M_{\sigma(1)}, M_{\sigma(2)}, \ldots, M_{\sigma(k)}$, where σ is a permutation on the set $\{1, 2, \ldots, k\}$. The commutation rule (3) implies that only the relative position of M_i with respect to $M_{i\pm1}$ matters. A simple word of length k can therefore be written as $W_k(s_2, s_3, \ldots, s_k)$ where the Boolean variable s_j for $2 \leqslant j \leqslant k$ is defined as follows: $s_j = 0$ if M_j is on the left of M_{j-1} and $s_j = 1$ if M_j is on the right of M_{j-1}. Equivalently, $W_k(s_2, s_3, \ldots, s_k)$ is uniquely defined by the recursion relation

$$W_k(s_2, s_3, \ldots, s_{k-1}, 1) = W_{k-1}(s_2, s_3, \ldots, s_{k-1})M_k, \tag{4}$$

$$W_k(s_2, s_3, \ldots, s_{k-1}, 0) = M_k W_{k-1}(s_2, s_3, \ldots, s_{k-1}). \tag{5}$$

The set of the 2^{k-1} simple words of length k will be called $\mathcal{W}_k$. For a simple word W_k, we define $u(W_k)$ to be the number of *inversions* in W_k, i.e., the number of times that M_j is on the left of M_{j-1}:

$$u(W_k(s_2, s_3, \ldots, s_k)) = \sum_{j=2}^{k} (1 - s_j). \tag{6}$$

We remark that simple words are connected, i.e., they cannot be factorized in two (or more) commuting words.

We introduce the ring ordered product $\mathcal{O}()$ which acts as follows on words of the type $W = M_{i_1} M_{i_2} \ldots M_{i_k}$, with $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant L$.

(i) If $i_1 > 1$ or $i_k < L$, we define $\mathcal{O}(W) = W$, i.e., W is well ordered.

(ii) If $i_1 = 1$ and $i_k = L$, we first write W as a product of two blocks, $W = AB$, such that $B = M_b M_{b+1} \cdots M_L$ is the maximal block of matrices with consecutive indices that contains M_L, and $A = M_1 M_{i_2} \cdots M_{i_a}$, with $i_a < b - 1$, contains the remaining terms. We then define

$$\mathcal{O}(W) = \mathcal{O}(AB) = BA = M_b M_{b+1} \cdots M_L M_1 M_{i_2} \cdots M_{i_a}. \tag{7}$$

(iii) The previous definition makes sense only for $k < L$. Indeed, when $k = L$, we have $W = M_1 M_2 \cdots M_L$ and it is not possible to split W in two different blocks A and B. For this special case, we define

$$\mathcal{O}(M_1 M_2 \cdots M_L) = |1, 1, \ldots, 1\rangle\langle 1, 1, \ldots, 1|. \tag{8}$$

3. A formula for the connected operators

The algebraic Bethe ansatz leads to a one-parameter commuting family of transfer matrices, $t(\lambda)$, containing the translation operator $T = t(1)$ and the Markov matrix $M = t'(0)$. This transfer matrix can be expanded in terms of non-local generalized Hamiltonians H_k, acting on the configuration space

$$t(\lambda) = 1 + \sum_{k=1}^{L} \lambda^k H_k. \tag{9}$$

(Note that $t(\lambda)$ was denoted by $t_g(\lambda)$ in Golinelli and Mallick (2007)). The operator H_k is a homogeneous sum of words of length k:

$$H_k = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant L} \mathcal{O}\big(M_{i_1} M_{i_2} \cdots M_{i_k}\big), \tag{10}$$

where $\mathcal{O}()$ is the ring-ordered product. Therefore, we have

$$t(\lambda) = \mathcal{O}\left(\prod_{i=1}^{L} (1 + \lambda M_i)\right). \tag{11}$$

We remark that at most $L - 1$ operators H_k have a non-trivial action. As usual, the local connected and extensive operators are obtained by taking the logarithm of the transfer matrix. For $k \geqslant 1$, the connected Hamiltonians F_k are defined as

$$\ln t(\lambda) = \sum_{k=1}^{\infty} \frac{\lambda^k}{k} F_k \qquad \text{or, equivalently} \qquad \sum_{k=1}^{\infty} \lambda^k F_k = \lambda t(\lambda)^{-1} t'(\lambda). \tag{12}$$

The second equation is obtained by taking the derivative of the first one with respect to λ and recalling that $t(\lambda)$ commutes with $t'(\lambda)$.

3.1. Elimination of the ring-ordered product

Expanding $t(\lambda)^{-1}$ with respect to λ, formula (12) allows us to calculate F_k as a polynomial function of the H_k's. By using (10), we observe that F_k is *a priori* a linear combination of products of k local operators M_i. However, this expression can be simplified by using the algebraic rules (1), (2) and (3) and *in fine*, F_k will be a linear combination of reduced words of length $j \leqslant k$. We know from formula (10) that at most $L - 1$ operators H_k are independent in a system of size L; we shall therefore calculate F_k only for $k \leqslant L - 1$. Thus, we need to consider reduced words of length $j \leqslant L - 1$.

Let W be such a word and $\mathcal{I}(W)$ be the set of indices of the operators M_i that compose W; our aim is to calculate its prefactor from equation (12). Because the rules (1) and (3) do not suppress or add any new index, the following property is true: if a word W' appearing in $\lambda t(\lambda)^{-1} t'(\lambda)$ is such that $\mathcal{I}(W') \neq \mathcal{I}(W)$ then even after simplification, W' will remain different from W. Therefore, the prefactor of W in $\sum \lambda^k F_k$ is the same as the prefactor of W in $\lambda t_{\mathcal{I}}(\lambda)^{-1} t'_{\mathcal{I}}(\lambda)$, where

$$t_{\mathcal{I}}(\lambda) = \mathcal{O}\left(\prod_{i \in \mathcal{I}} (1 + \lambda M_i)\right) \qquad \text{with} \quad \mathcal{I}(W) \subset \mathcal{I}. \tag{13}$$

Because F_k commutes with the translation operator T, then the prefactor of $W = M_{i_1} M_{i_2} \cdots M_{i_j}$ is the same as the prefactor of $T^r M T^{-r} = M_{r+i_1} M_{r+i_2} \cdots M_{r+i_j}$ for any $r = 1, \ldots, L - 1$. Furthermore, any word W of size $k \leqslant L - 1$ is equivalent, by a translation, to a word that contains M_1 and not M_L: indeed, there exists at least one index i_0 such that $i_0 \notin \mathcal{I}(W)$ and $(i_0 + 1) \in \mathcal{I}(W)$ and it is thus sufficient to translate W by $r = L - i_0$.

In conclusion, it suffices to study in expression (12) the reduced words W with set of indices included in $\mathcal{I}^* = \{1, 2, \ldots, L - 1\}$. Because the index L does not appear in $\mathcal{I}^*$, the ring-ordered product has a trivial action in equation (13) and we have

$$t_{\mathcal{I}^*}(\lambda) = (1 + \lambda M_1)(1 + \lambda M_2) \cdots (1 + \lambda M_{L-1}). \tag{14}$$

We have thus been able to eliminate the ring-ordered product using the TASEP algebra and the translation operator.

3.2. Calculation of the connected operators

Differentiating $t_{\mathcal{I}^*}(\lambda)$ with respect to λ in equation (14), we have

$$t'_{\mathcal{I}^*}(\lambda) = \sum_{i=1}^{L-1} (1+\lambda M_1)\cdots(1+\lambda M_{i-1})M_i(1+\lambda M_{i+1})\cdots(1+\lambda M_{L-1}). \quad (15)$$

Using relation (1), we deduce that for any $\lambda \neq 1$:

$$(1+\lambda M_i)^{-1} = (1+\alpha M_i) \qquad \text{with} \quad \alpha = \frac{\lambda}{\lambda - 1}. \quad (16)$$

Therefore, we have

$$t_{\mathcal{I}^*}(\lambda)^{-1} = (1+\alpha M_{L-1})(1+\alpha M_{L-2})\cdots(1+\alpha M_1). \quad (17)$$

Noting that $\lambda(1+\alpha M_i)M_i = -\alpha M_i$, we deduce

$$\lambda t_{\mathcal{I}^*}(\lambda)^{-1} t'_{\mathcal{I}^*}(\lambda) = -\alpha \sum_{i=1}^{L-1} (1+\alpha M_{L-1})\cdots(1+\alpha M_{i+1})M_i(1+\lambda M_{i+1})\cdots(1+\lambda M_{L-1}). \quad (18)$$

The ith term in this sum contains words with indices between i and $L-1$. Because we are looking for the words that contain the operator M_1, we must consider only the first term in this sum, which we note by Q:

$$Q = -\alpha(1+\alpha M_{L-1})\cdots(1+\alpha M_2)M_1(1+\lambda M_2)\cdots(1+\lambda M_{L-1}). \quad (19)$$

In the appendix, we show that

$$Q = R_1 + R_2 + \cdots + R_{L-1}, \quad (20)$$

where R_i is defined by the recursion

$$R_1 = -\alpha M_1, \quad (21)$$

$$R_i = \lambda R_{i-1} M_i + \alpha M_i R_{i-1} \qquad \text{for} \quad i \geqslant 2. \quad (22)$$

To summarize, all the words in $\sum_{k=1}^{\infty} \lambda^k F_k$ that contain M_1 and not M_L are given by $Q = R_1 + R_2 + \cdots + R_{L-1}$. From the recursion relation (22), we deduce that R_i is a linear combination of the 2^{i-1} simple words $W_i(s_2, s_3, \ldots, s_i)$ defined in equations (4), (5). Furthermore, we observe from (22) that a factor λ appears if $s_i = 1$ and a factor $\alpha = \lambda/(\lambda-1)$ appears if $s_i = 0$. Therefore, the coefficient $f(W)$ of $W = W_i(s_2, s_3, \ldots, s_i)$ in Q is given by

$$f(W) = (-1)^u \frac{\lambda^i}{(1-\lambda)^{u+1}} = (-1)^u \sum_{j=0}^{\infty} \binom{u+j}{j} \lambda^{i+j}, \quad (23)$$

where i is the length of W and $u = u(W)$ is its inversion number, defined in equation (6). We have thus shown that

$$Q = \sum_{i=1}^{L-1} \sum_{W \in \mathcal{W}_i} f(W) W = \sum_{i=1}^{L-1} \sum_{W \in \mathcal{W}_i} W \sum_{j=0}^{\infty} (-1)^{u(W)} \binom{u(W)+j}{j} \lambda^{i+j}, \quad (24)$$

where $\mathcal{W}_i$ is the set of simple words of length i.

Finally, we recall that the coefficient in $\sum_{k=1}^{\infty} \lambda^k F_k$ of a reduced word W that contains M_1 and not M_L is the same as its coefficient in Q. Extracting the term of order λ^k in equation (24), we deduce that any word W in F_k that contains M_1 and not M_L is a simple word of length $i \leqslant k$ and its prefactor is given by $(-1)^{u(W)}\binom{u(W)+k-i}{k-i}$.

The full expression of F_k is obtained by applying the translation operator to the expression (24); indeed any word in F_k can be uniquely obtained by translating a simple word in F_k that contains M_1 and not M_L. We conclude that for $k < L$,

$$F_k = \mathcal{T} \sum_{i=1}^{k} \sum_{W \in \mathcal{W}_i} (-1)^{u(W)} \binom{k-i+u(W)}{k-i} W, \tag{25}$$

where $\mathcal{T}$ is the translation symmetrizator that acts on any operator A as follows: $\mathcal{T}A = \sum_{i=0}^{L-1} T^i A T^{-i}$. The presence of $\mathcal{T}$ in equation (25) insures that F_k is invariant by translation on the periodic system of size L. All simple words being connected; formula (25) implies that F_k is connected.

4. Conclusion

The exact combinatorial expression (25) for the connected operators that commute with the TASEP Markov matrix fully elucidates the hierarchical structure derived from the algebraic Bethe ansatz. It would be of a great interest to extend this result to the partially asymmetric exclusion process (PASEP) in which a particle can make forward and backward jumps with probabilities p and q, respectively. In particular, we recall that the symmetric exclusion process is equivalent to the Heisenberg spin chain: in this case the connected operators have been calculated for the lowest orders (Fabricius *et al* 1990). This is a challenging and difficult problem. In our derivation we used a fundamental property of the TASEP algebra: rules (1)–(3) when applied to a word W either cancel W or conserve the set of indices $\mathcal{I}(W)$. However, the PASEP algebra associated violates this crucial property because there we have $M_i M_{i+1} M_i = pq M_i$. Therefore the method followed here does not have a straightforward extension to the PASEP case.

Appendix. Proof of equation (20)

Let us define the following series:

$$Q_1 = -\alpha M_1, \tag{A.1}$$

$$Q_i = (1 + \alpha M_i) Q_{i-1} (1 + \lambda M_i) \qquad \text{for} \quad i \geqslant 2, \tag{A.2}$$

so that Q defined in equation (19) is given by $Q = Q_{L-1}$. Let us consider R_i defined by the recursion relation (22). The indices that appear in the words of Q_i and R_i belong to $\{1, 2, \ldots, i\}$. Therefore, we have

$$[R_j, M_i] = 0 \qquad \text{for} \quad j \leqslant i - 2, \tag{A.3}$$

because the operators $M_1, M_2, \ldots, M_j$ that compose R_j commute with M_i. From equations (A.3) and (16), we obtain

$$(1 + \alpha M_i) R_j (1 + \lambda M_i) = R_j \qquad \text{for} \quad j \leqslant i - 2. \tag{A.4}$$

Furthermore, from (22), we obtain

$$M_i R_{i-1} M_i = \lambda M_i R_{i-2} M_{i-1} M_i + \alpha M_i M_{i-1} R_{i-2} M_i. \tag{A.5}$$

Because M_i commutes with R_{i-2}, we deduce from $M_i M_{i-1} M_i = 0$ that

$$M_i R_{i-1} M_i = 0. \tag{A.6}$$

Using equation (A.6), we find

$$(1 + \alpha M_i) R_{i-1} (1 + \lambda M_i) = R_{i-1} + \lambda R_{i-1} M_i + \alpha M_i R_{i-1} = R_{i-1} + R_i. \tag{A.7}$$

From equations (A.4) and (A.7), we find that the unique solution of the recursion relation (A.2) is given by $Q_i = R_1 + R_2 + \cdots + R_i$. Then $Q = Q_{L-1}$ is given by equation (20).

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